For many years, as a high school mathematics teacher, I thought of the Pythagorean theorem—\(a^2 + b^2 = c^2\)—as simply an algebraic tool to find a missing side of a right triangle. Numbers were to be plugged into the formula, and the unknown was to be solved for algebraically. The formula had to wait to be introduced until students knew about square roots, which were presented as a numerical concept, totally divorced from their physical aspect. Students also needed to know how to solve equations containing square roots, a skill usually introduced near the end of algebra 1.

In geometry, both when I was a student and when I began teaching it, the physical attributes of the right triangle were discussed. However, squaring a number was taught as the operation of multiplying a number times itself. The idea of squaring a number by physically forming a square from a length representing that number and finding the area of the resulting square was never explicitly taught.

After a long career as a high school mathematics teacher, I became the principal of a school for grades pre-K–5 in rural Vermont. I began working with groups of five to six students from each grade level once a week for mathematics enrichment or remediation. The students enjoyed it, and I loved getting back into mathematics and learning what young children are capable of mathematically. Having no preconceived notions, no set curriculum, and no time restraints, I posed problems for the students and let the learning happen. Although I introduced the unit that I present in this article to a group of sixth-grade students, I have since successfully presented lessons within this unit to entire fourth- and fifth-grade classes.

I filled my office with mathematics manipulatives and experimented, although I had rarely used manipulatives when teaching high school mathematics. I pulled out the geoboards and first used them to give the students practice in visualizing symmetry. All my groups from grades 1 through 5 were able to construct symmetric shapes across a line (see fig. 1). With the fourth and fifth graders, I started exploring the areas of polygons. Given the length of one side of a small square on the geoboard as a unit, the students were quickly able to find the areas of rectangles. They discovered that they could find the area of a triangle by drawing a rectangle around it and seeing that it was half of the rectangle. They weren’t using a formula; they were using their intuition.

The next logical step was finding perimeters. Rectangles were no problem, but once we looked at a triangle or any polygon involving a diagonal, there was not always a natural number to describe the length. The first time the students told me that the perimeter of a right triangle whose legs had a length of 1 unit was 3, I pulled out the compasses. I had them open the compasses to a length that matched the two sides having the length of 1 unit and then, without moving the width of the compass, place it on the hypotenuse (see fig. 2). The students could clearly see that the length of the hypotenuse was more than 1, so they started making guesses—1.5, 1.25, and so forth. They narrowed the length down to between 1.25 and 1.5. They were close, I told them, but there was a name for the length, and we would figure it out.

By Elaine K. Watson

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Definitions: A Place to Start

The next time I met with the students, I told them about the Greek philosopher and mathematician named Pythagoras, who was born in 569 B.C.E. After a discussion that included the meaning of the abbreviation B.C.E., a number line, and negative numbers, the students figured out how many years ago Pythagoras was born.

I asked a student to look up the Pythagorean theorem in the dictionary. The definition, according to Merriam-Webster’s Dictionary (1998 paper), is as follows: “A theorem in geometry: the square of the length of the hypotenuse of a right triangle equals the sum of the squares of the lengths of the other two sides.” I had a student write this definition on the board. After reading the definition, the students weren’t sure what hypotenuse meant. Another student looked up hypotenuse in the same dictionary and found this definition: “the side of a triangle having a right angle that is opposite the right angle.” Awkward though this definition was, we muddled through it, drew a few right triangles on the board, and made sure that everyone was clear about what the hypotenuse is.

Now back to the Pythagorean theorem and muddling through that definition: “the square of the length of the hypotenuse.” What did that mean? The students knew what a square is, but what is the square of the length of the hypotenuse? I skipped that part of the definition and went to the last part: “the squares of the lengths of the other two sides.” I hinted that this must mean the square formed when the side of a triangle was one side of the square. I asked a student to sketch a square on one side of the triangle that wasn’t the hypotenuse. This seemed to be easier because we had drawn all the right triangles with the two legs horizontal and vertical, making the hypotenuse sloped. Because of the placement of the triangles we had drawn, visualizing a square connected to the two legs was easier than visualizing the square connected to the hypotenuse. After a few sketches, the students understood what was meant by “the square of the length of the hypotenuse” and “the squares of the lengths of the other two sides.” I then added a couple of right triangles whose legs were not horizontal and vertical to make sure that the students could still visualize the squares formed by all three sides. They were successful, so now it was time to try to make sense of the entire definition.

We went through every word. We decided that in order to talk about the definition, it would be a good idea to give each side of the triangle a name. The students came up with three letters (from the first letters of some of their names), so we had $t$, $m$, and $e$, with $m$ being the hypotenuse. Then we needed to name the squares formed on each of these sides. The group decided that it would make sense to name them the capital letter matching the name of the side—$T$, $M$, and $E$—with the square formed off each side having the capital letter matching the name of the side. Back to the Merriam-Webster’s definition: “The square of the length of the hypotenuse of a right triangle $[M]$ equals $[=]$ the sum of $[after some discussion, we agreed that this meant to add]$ the squares of the lengths of the other two sides $[T and E].”$ Translating the word sentence into a number sentence, we wrote $M = T + E$. How can one square equal the sum of two other squares? Merriam-Webster’s must mean the area of the square of the hypotenuse is equal to the sum of the areas of the squares of the other two sides.

I gave the students an easy example—a 3, 4, 5...
right triangle (i.e., a right triangle with sides measuring 3, 4, and 5 units, respectively). I had them draw a right triangle with legs of 3 units and 4 units and sketch in the hypotenuse without giving them the length (see fig. 3). I asked them to draw on graph paper the squares off each leg, which they could easily do and see that the areas were \( T = 9 \) square units and \( E = 16 \) square units. I asked, “If the Pythagorean theorem is true, what should the area of the square off the hypotenuse equal?” The students looked at the formula \( M = T + E \) written on the board. They then substituted 9 for \( T \) and 16 for \( E \), added 9 + 16, and found that \( M = 25 \) (square units). I asked them to sketch the square off the hypotenuse. This is not an easy task to accomplish accurately without considering slope. At this point, I had to decide how rigorous I wanted to be. Did I want to take a long detour and have the students prove that the area is 25? Would doing this take such a detour that they would lose sight of our original task—to find a way to accurately measure the perimeter of triangles or rectangles on the geoboard? I chose not to at the time. Instead, I had the students label the area \( M = 25 \) square units and asked them what length they thought the length of the hypotenuse \( m \) would be if the square \( M \) formed by the hypotenuse had an area of 25 square units. In relatively short order, they deduced that the length of \( m \) was 5 units. The students had now seen the Pythagorean theorem in action with nice natural numbers. The next step was to confound them before I sent them away to think.

**Disequilibrium**

*Equilibrium* is defined as “a state of intellectual or emotional balance,” according to *Merriam-Webster’s Medical Dictionary* (2002), so *disequilibrium* is the absence of this balance. I believe that students become more engaged with problems and learn how to solve problems more effectively when the teacher allows them to wallow in a state of disequilibrium for awhile.

To put the students a little off-balance intellectually, I sketched an isosceles right triangle with the legs each a length of 1 unit and asked them to construct this triangle on the geoboard. I did not label the length of the hypotenuse. The students repeated on the geoboard the steps we had done on the previous 3, 4, 5 triangle: They formed the squares off the leg, found that the areas of each square was 1 square unit, plugged those numbers into their formula \( M = T + E \), and knew that the area of the square \( M \), the one off the hypotenuse, was 2 square units. Now for the disequilibrium: What is the length of the hypotenuse? We had already shown that it was more than 1. What number times itself equals 2? The students started guessing. 1.5? We used a calculator to check 1.5 × 1.5 and got 2.25. How about 1.25? 1.25 × 1.25 = 1.5625. They said, “It’s impossible! There is no such number!” I said, “But there must be. We see a length, and that length must represent a number.” I sent them away until the following week, wondering and off-balance.

Over the next few days, a couple of students approached me with what they thought was the answer. One student excitedly told me that the answer was –1. I showed him on the calculator how \((-1) \times (-1) = +1\). Another student claimed that he had found the answer on the Internet: 1.414. I said that he was awfully close to the real number. In fact, if we multiply 1.414 by 1.414, we get 1.999396, which is very close to 2, so he was on the right track. I gave him a hint: 1.414 was an approximation of the number we were looking for, but the number we were looking for had a name and was exact.

The week passed, and the students continued to hound me to tell them the answer. I wouldn’t budge, but I did tell one of the teachers what we were up to. She let the answer slip, and by the time the students
came back the next week, they knew that the number I was looking for was $\sqrt{2}$.

I filled them in on some mathematical history. The Pythagoreans, Pythagoras’s students, were sworn to secrecy on the discovery of this kind of number, which could not be expressed as a ratio of two other numbers. They called it *incommensurate*. We now call it *irrational*. We can see that $\sqrt{2}$ exists because it is the length of a line. However, until that length was given a proper name, we could only estimate it with decimal approximations such as 1.414. Of course, any time we draw something in mathematics, our drawings will never be totally exact. It is the *idea* of exactness that we are striving for. The exact length is $\sqrt{2}$, which can be approximated as 1.414.

**Finding Other Square Roots**

The next lesson was spent discovering other irrational lengths on the geoboard. We built right triangles with legs of 1 and 2 units and constructed the square figures off all three sides. Reviewing the Pythagorean theorem, the students deduced that the square off the hypotenuse would be $1 + 4 = 5$. So what number times itself would equal 5? Looking back at the previous example of the 1, 1, $\sqrt{2}$ triangle, they guessed correctly that it would be $\sqrt{5}$. I let the students loose to find the lengths of the hypotenuses in other right triangles that they could form on the geoboard. When the square constructed on the hypotenuse would not fit on the board, we placed another geoboard adjacent to the original. After a while, the students got to where they didn’t need to construct the square for the hypotenuse but still needed to construct the square of the measurement of the other two legs so that they could see the squares. Eventually, many of the students could visualize the squares of the two legs and did not have to construct these to find the length of the hypotenuse.

**Square Root Ruler**

The next time I saw the students, I pretended that I was Pythagoras and they were the Pythagoreans. I was tired of having to think about squares so much, I said, and I needed a handy tool to measure lengths that were square roots. What I needed was a square root ruler that I could carry with me. Most of the square roots I dealt with were not very large. In fact, a ruler with all square roots from $\sqrt{1}$ to $\sqrt{10}$ was all that I needed for now.

In preparation, I had taken a standard-size (8 1/2-
by-11-inch) page of half-inch-square graph paper and zoomed in on the copier to enlarge the image to fit an 11-by-17-inch sheet. The resulting graph paper had squares that measured approximately one square inch in area. Using the larger grid would provide students with a longer number line on which to place ten square roots over a length that (as they would discover) would cover only a little more than 3 units on the scale. The tools I gave them to make the square root ruler were a few sheets of the enlarged 11-by-17-inch graph paper, a straight edge, and a compass to measure and copy lengths as we had done earlier with the $1, 1, \sqrt{2}$ triangle (see fig. 2).

With the skills and the tools ready to make the ruler, I suggested that we use the unit represented by the grid on the graph paper as equal to a length of 1 unit. We could create the ruler near the bottom of the 11-by-17-inch sheet. Would it fit? How many units long would it need to be to accommodate all

\begin{figure}
\centering
\includegraphics[width=\textwidth]{square_root_ruler}
\caption{Student work resulting from investigating $\sqrt{3}$ to create the square root ruler}
\end{figure}

\begin{enumerate}
\item For the “ruler,” draw a horizontal line across the bottom of the graph paper, leaving room below the line for labels. The line needs to be only about 3.5 units. Label the integers 0, 1, 2, and 3 on the ruler.
\item After discussion, label $\sqrt{1}$ below the integer 1 on the ruler.
\item After discussion, draw a $1, 1, \sqrt{2}$ triangle. Measure the hypotenuse of this triangle with the compass. Without changing the opening of the compass, put the point at zero on the ruler, mark off the length $\sqrt{2}$ on the ruler, and label it.
\item After discussion about how to find $\sqrt{3}$, form a triangle whose legs are $\sqrt{1}$ and $\sqrt{2}$. $\sqrt{1} = 1$ and $\sqrt{2}$ is the current length of the compass opening. In this new right triangle, the squares formed by the legs will have areas of 1 and 2. According to the Pythagorean theorem, the square of the hypotenuse equals the sum of the squares of the legs, so the square of the hypotenuse will be $1 + 2 = 3$. Therefore, the hypotenuse will have length $\sqrt{3}$ units.
\item Use the compass to measure the length $\sqrt{3}$ units. Without changing the width of the compass opening, put the point on zero on the ruler and copy the length $\sqrt{3}$ onto the ruler and label.
\end{enumerate}

Note: Some students may figure out that they don’t have to build a triangle to find $\sqrt{4}$ and $\sqrt{9}$. 
the square roots from $\sqrt{1}$ to $\sqrt{10}$? The students’ initial answer was that the ruler would need to be 10 units long. I gave them a hint: Because $10 \times 10 = 100$, then $\sqrt{100} = 10$, so surely the ruler wouldn’t need to be that long. What number times itself is near 10? After a few guesses, the students came up with $3 \times 3 = 9$, which meant that the square root of 10 was a little more than 3, so they determined that the ruler would need to be only a little longer than 3 units (see fig. 4).

We started together. For the ruler, each student drew a horizontal pencil line about $3\frac{1}{2}$ units long near the bottom of the graph paper, leaving room below the line to clearly label the lengths. What is the square root of 1? Hmm … $1/2$? $1/4$? After a few wrong guesses, the students realized that $1^{1} = 1$, so they marked that on the pencil line and labeled it 1 (see fig. 5, steps 1 and 2).

How about the square root of 2? With their experience with the 1, 1, $\sqrt{2}$ triangle, the students drew it, copied the length of the hypotenuse onto the ruler by using a compass, and labeled it $\sqrt{2}$. Easy, right? (See fig. 5, step 3.)

How about the square root of 3? How could the students construct that? The Pythagorean theorem told them that they needed to have squares on each leg whose areas would add up to 3 square units. An obvious choice would be 1 + 2 = 3. An area of 1 was easy, but what kind of square has an area of 2 square units? Oh! That’s the length of the hypotenuse in the previous triangle! A square with sides equal to the square root of 2 ($\sqrt{2}$) would have an area of 2 square units. So, using the compass to measure the length of the hypotenuse of the 1, 1, $\sqrt{2}$ triangle, we copied the length $\sqrt{2}$ along the grid line forming a right angle with the leg of the 1-unit length. The length of the hypotenuse connecting them is $\sqrt{3}$! We copied that length onto the square root ruler and labeled it. (See fig. 5, steps 4 and 5.)

I then let the students work by themselves. Some of them realized before they constructed the triangles that the square roots of 4 and 9 were 2 and 3, respectively, and simply marked these numbers on the ruler. If they didn’t realize this, I let them discover it on their own. I supplied the students with as much of the 11-by-17-inch graph paper as needed. Most used two or three pages.

To wrap up the lesson and bring it back to familiar numbers, I asked the students to guess the decimal estimate of the lengths of $\sqrt{1}$ to $\sqrt{10}$ to the nearest tenth simply by looking at the square root ruler. Then I gave them a scientific calculator and had them enter the square roots to see what the calculator told them. I asked them to write the calculator answer to three decimal places. All the students’ estimations were extremely close. A copy of one student’s table can be seen in figure 6.

### Back to the Geoboard to Find Areas and Perimeters

Our concluding lesson went back to working with many-sided convex and concave polygons. Now, in addition to finding the area, the students had the skills necessary to find the perimeter as well. First, they needed to learn how to think of square roots as normal numbers and learn how to calculate with them. I started with a 1, 1, $\sqrt{2}$ right triangle and asked for the area and the perimeter. For the perimeter, we needed to add 1 + 1 + $\sqrt{2}$. I explained that this measure could be expressed as $2 + \sqrt{2}$ units; we could combine the 1 and 1, but we couldn’t combine it to $\sqrt{2}$ because 2 and $\sqrt{2}$ were different types of numbers, yet $\sqrt{2}$ needed to be in the answer. I explained that we could substitute 1.414 for $\sqrt{2}$ and have the perimeter be $2 + 1.414 = 3.414$ units, but this answer was only a decimal approximation. The exact answer was $2 + \sqrt{2}$. We then formed a right triangle with legs having the length of 1 and 2 units. Using the knowledge they had gained throughout the unit, the students knew that the square off the

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**Figure 6**

Student record comparing estimates of square roots to results generated by using a scientific calculator

<table>
<thead>
<tr>
<th>Square root</th>
<th>Student decimal estimation</th>
<th>Calculator decimal estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{1}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>1.4</td>
<td>1.414</td>
</tr>
<tr>
<td>$\sqrt{3}$</td>
<td>1.7</td>
<td>1.732</td>
</tr>
<tr>
<td>$\sqrt{4}$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\sqrt{5}$</td>
<td>2.2</td>
<td>2.236</td>
</tr>
<tr>
<td>$\sqrt{6}$</td>
<td>2.4</td>
<td>2.449</td>
</tr>
<tr>
<td>$\sqrt{7}$</td>
<td>2.6</td>
<td>2.645</td>
</tr>
<tr>
<td>$\sqrt{8}$</td>
<td>2.8</td>
<td>2.828</td>
</tr>
<tr>
<td>$\sqrt{9}$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\sqrt{10}$</td>
<td>3.2</td>
<td>3.162</td>
</tr>
</tbody>
</table>
hypotenuse would have an area of $1 + 4 = 5$ square units, so the hypotenuse would have a length of $\sqrt{5}$ units. Therefore, the perimeter would be $1 + 2 + \sqrt{5} = 3 + \sqrt{5}$ units (see fig. 7). The area would be 1 square unit, since the triangle would be half a rectangle with an area of 2 square units. The students were pros at area, having mastered that skill earlier in the unit.

**Conclusion**

I did not plan this unit ahead of time. The lesson evolved according to challenges and questions generated by the students. My original goal was to “play” with the geoboards and see what would unfold. I started with symmetry and found that a wealth of visualization skills emerged as students from grades K through 5 challenged one another with shapes of increasing complexity to mirror across the line of symmetry. The next logical explorations on the geoboard were area and perimeter. I decided to limit the exploration of perimeter to fifth graders, the highest grade in my school, after I became aware of the special cases involving square roots that arose when trying to find the perimeter of triangles on the geoboard grid. I believe that second and third graders can understand the area part of the lesson, especially if the figures are kept fairly simple. Fourth graders can certainly grasp finding the areas of fairly complex polygons, and some will be able to interpret the Pythagorean theorem to find the perimeters.

The intertwining topics that fell out of the lessons amazed me. Area, perimeter, and symmetry would obviously be part of the conversation, but I was not expecting the lessons to extend into the concept of the Pythagorean theorem and its history, irrational numbers, like and unlike terms, compass measurements, and approximation and rounding. The students used no algebraic manipulations or formulas but were able to accurately find the areas and perimeters of many types of shapes simply by using their intuition. When, in the future, they learn the area formula for a triangle, $A = \frac{1}{2}bh$, my hope is that they will remember our work and know that $bh$ is the area of the rectangle that is twice the area of the given triangle.

I hope that this experience has taught these students that formulas can make sense intuitively; those they can visualize, they don’t have to memorize. The Pythagorean theorem is not simply an algebraic formula for finding the missing side of a right triangle. When these students are introduced to square roots in middle school mathematics, they will have a working knowledge of what they mean, that they are the length of a side of a square whose area is under the square root symbol. When they are reintroduced to the Pythagorean theorem in high school, they will recall that $a^2 + b^2 = c^2$ can be visualized as the squares formed off each side of a right triangle.

In my three years as the principal of an elementary school, I have been overwhelmed by students’ natural curiosity and their enthusiasm for exploring mathematics concepts. Elementary school students can comprehend numerous advanced mathematics concepts that are usually saved for when the students are older and learn algebra. By *algebra* I mean its traditional definition: performing mathematical operations on symbols represented by letters. Elementary school students can understand more advanced topics if we broaden the definition of the term to mean the study of patterns in mathematics. At this age students have not been jaded by the commonly held belief that mathematics is difficult and, therefore, not fun. Elementary school students can grasp abstract concepts if their instruction includes a systematic approach that starts with the use of concrete manipulatives, continues with asking the right questions to lead them to discover patterns, and finally allows them to discover how those patterns coalesce into a general rule that they can visualize and verbalize. Yes, this process takes longer, but the sparkle in my students’ eyes when they “get it” and the depth of understanding that enables them to “hold onto it” is well worth the effort.